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Rotating Frames in Special Relativity Analyzed in Light of a Recent Article by M. Strauss

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Abstract

The argument of Einstein for non-Euclidity on a rotating disk is analyzed and found valid. The kinematic reason for the non-Euclidean geometry is stated explicitly and provides a kinematic resolution of Ehrenfest's paradox. The transformation from an inertial frame K to a rotating frame, the axis of which is at rest in K, is discussed. It is concluded in favor of the Galilean-like transformation employed by Møller. The method used by Møller in obtaining the intrinsic spatial geometry in any frame is examined. It is found to be adequate, provided that only coordinates with a proper metrical significance are used. In this connection the distinction between global and local geometry is found to be essential.

1. *Introduction*

Recently Strauss (1974) has given a description of rotating frames in special relativity. In this work the important problem of selecting the most suitable transformation from an inertial system K to a rotating frame K', the axis of which is at rest in K, is investigated. One solution of this problem is to adopt a transformation with a Galilean character (Møller, 1952). The physical reason for this choice is that angular velocity, as opposed to translational velocity, is a quantity with an absolute value that can be locally measured, both mechanically (by use of Fouceault's pendulum) and optically [Sagnac's experiment (Post, 1967)] (Grøn, 1975). Wanting, however, a transformation that has the Lorentz transformation between K and an inertial observer instantaneously at rest relative to a point in K' as limit for small angular velocity ω and large radius r, such that the uniform rotation tends to a uniform translation, Franklin (1922), Trocheries (1949), and Takeno (1952) have proposed a Lorentz-like transformation with $\omega r/c$ as velocity parameter (Taylor and Wheeler, 1966). Strauss, on the other hand, has proposed a Lorentz-like transformation with ωr as velocity.

He has also criticized Einstein's argument for non-Euclidity on a rotating disk and the method used by Møller (1952) in obtaining the intrinsic geometry in any frame.

In this work the argument of Einstein, mentioned above, the method used by Møller, and the transformations to a rotating frame are discussed in light of the recent article by Strauss. It is concluded in favor of the Einstein argument and the Galilean-like transformation. The method used by Møller in obtaining the intrinsic geometry in any frame is found to be adequate, provided that only coordinates with a proper metrical significance are used. In this connection the distinction between global and local geometry is found to be essential. In inertial frames the relevant coordinates, both globally and locally, are the Minkowskian ones, implying that different inertial systems are connected with Lorentz transformations.

2. *On Einstein's Argument for Non-Euclidity on a Rotating Disk*

The argument of Einstein (1921) is as follows:

> Let K' be a system of coordinates whose z'-axis coincides with the z-axis of [an inertial system] K, and which rotates about the latter axis with constant angular velocity. ··· Imagine a circle drawn about the origin in the $x'y'$-plane of K', and a diameter of this circle. Imagine further that we have given a large number of rigid rods, all equal to each other. We suppose these laid in series along the periphery and the diameter of the circle, at rest relatively to K'. If U is the number of these rods along the periphery, D the number along the diameter, then, if K' does not rotate relatively to K, we shall have $U/D = \pi$. But if K' rotates we get a different result. Suppose that at a definite time t of K we determine the ends of all the rods. With respect to K all the rods upon the periphery experience the Lorentz contraction, but the rods upon the diameter do not experience this contraction (along their lengths!).[1] It therefore follows that $U/D > \pi$. It therefore follows that the laws of configuration of rigid bodies with respect to K' do not agree with the laws of configuration of rigid bodies that are in accordance with Euclidean geometry.

According to the view of Strauss this argument is wrong, and he writes

> If the measuring rods laid along the circumference of the rotating disk are Lorentz contracted with respect to the inertial frame, so are the distances on the circumference they are supposed to measure; hence the two effects would cancel each other, and the ratio U/D would turn out to equal π as in the Euclidean plane.

The question of whether both the measuring rods laid along the circumference of the rotating disk and the elements of the circumference do have a Lorentz contraction, will now be investigated, using special relativity.

As pointed out by Møller (1952), if measuring rods are kept in a fixed position relative to an accelerated system of reference, they will generally be submitted to forces that may cause a deformation of the measuring rods. This

[1] These considerations assume that the behavior of rods and clocks depends only upon velocities, and not upon accelerations, or, at least, that the influence of acceleration does not counteract that of velocity.

deformation will, however, depend on the elastic properties of the material from which the measuring rods are made, and all such deformations of the measuring rods can therefore be corrected for. In general it is assumed that the (corrected) *standard* measuring rods in an accelerated system are subjected to Lorentz contractions only, which means that the lengths of the rods are independent of the accelerations. Thus all standard measuring rods are assumed to perform Born rigid motions, so that their proper lenghts are unchanged (Newburgh, 1974).

In order to obtain this, n rods are assumed to rest on the disk without friction, being kept in place by a frictionless rim on the circumference of the disk, each rod being fastened to the disk at one end only, at points P_k, so that they just cover the circumference when the disk is not rotating, as shown in Figure 1.

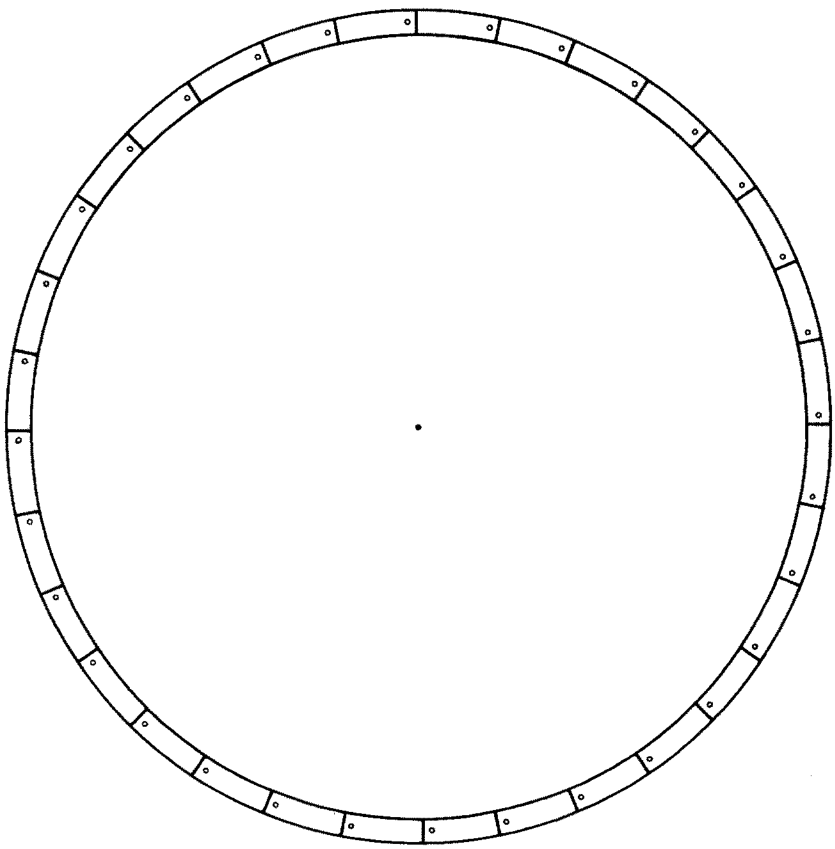

Figure 1. The disk and the measuring rods at rest.

Now we regard the process of accelerating the disk with the rods, so that it gets an angular velocity. At the moment considered the disk has an angular velocity ω, which is to be increased. The (corrected) acceleration of the rods and the disk must be prescribed so that (a) the proper length L_0 of the rods remains unchanged, and (b) no kinematic inconsistencies result.

Condition (a) demands that in the instantaneous rest frame K_k of each rod, every point of the rod with which this intertial frame is associated is accelerated simultaneously, when elastic deformations and delays are corrected for. According to the Lorentz transformations from K_k to K one observes in K that the front of each rod is accelerated at a time $(\omega r/c^2)L_0$ later than the rear end of it. Thus each rod gets an increased Lorentz contraction due to the acceleration. When the disk has an angular velocity ω, every rod is observed in K with a length $L = L_0(1 - \omega^2 r^2/c^2)^{1/2}$.

According to the view of Strauss the circumference of the disk is Lorentz contracted in the same way. This will here be investigated from a kinematic point of view.

Now we associate an instantaneous inertial rest frame E_k with each element between two neighboring points P_k and P_{k+1} on the circumference. If the accelerations of each set of neighboring points are simultaneous, as measured in the associated rest frame, so that the proper lengths of the elements remain unchanged, the circumference is Lorentz contracted as observed in K. However, it may be shown (Grøn, 1975) that it is kinematically self-contradicting to assume that all n points on the circumference get accelerations simultaneously as measured in the successive inertial systems E_k. Thus a transition of the disk from rest to rotational motion, while it satisfies Born's definition of rigidity, is a kinematical impossibility. In this way the relativistic kinematics leads to the conclusion that the circumference of a rotating disk is not Lorentz contracted.

The only isotropic way of giving the disk an angular velocity is to accelerate all points P_k simultaneously as measured in K. In E_k one then measures that the point P_k is accelerated at a point of time

$$\Delta t_k = (1 - \omega^2 r^2/c^2)^{-1/2}(\omega r/c^2)\, L_0 \tag{2.1}$$

earlier than the point P_{k+1}. Thus the distance between these points, that is the point at the front of one measuring rod and at the front of the next, increases, as observed in E_k. However, as the (corrected) measuring rods are moving rigidly, their proper lengths remain unchanged. Accordingly the rods separate from each other as the disk accelerates. If the velocity of a rod is increased from $r\omega$ to $r(\omega + d\omega)$ as observed in K, its velocity change, as observed from E_k, is $(1 - \omega^2 r^2/c^2)^{-1} r\, d\omega$. During this change the distance between two neighboring rods increases with

$$ds_k = (1 - \omega^2 r^2/c^2)^{-3/2}(\omega r^2/c^2) L_0\, d\omega \tag{2.2}$$

Integrating, one finds the distance between the rods, as measured in E_k, when the disk rotates with an angular velocity ω:

$$s_k = [(1 - \omega^2 r^2/c^2)^{-1/2} - 1] L_0 \tag{2.3}$$

Thus the distance as measured in K is

$$s = L_0 - L_0(1 - \omega^2 r^2/c^2)^{1/2} \tag{2.4}$$

in accordance with the fact that the measuring rods are Lorentz contracted, while the circumference of the disk is not. The observation in K of the rotating disk and the measuring rods is shown in Figure 2.

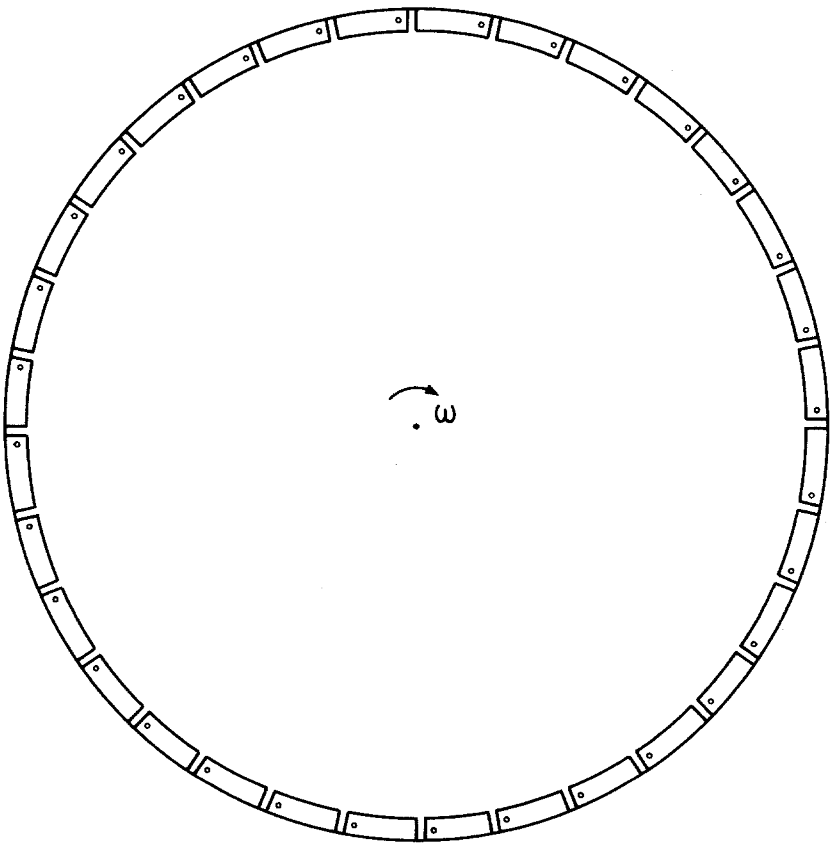

Figure 2. The rotating disk and the measuring rods.

The results of the above analysis imply that the proportion between the length of the circumference of a rotating disk and the length of its diameter, measured with (corrected) standard measuring rods, rotating with the disk, is

$$f = (1 - \omega^2 r^2/c^2)^{1/2} \tag{2.5}$$

Since this statement is invariant under a transformation connecting two different coordinate systems inside the same system of reference, as it is based on the use of standard instruments, it may be regarded as a statement characterizing the intrinsic spatial geometry on the rotating disk. It follows from equation (2.5) that this geometry is non-Euclidean.

3. *Coordinate Transformations to a Rotating Frame*

In selecting a coordinate system for a rotating frame, one should aim at obtaining an unambiguous kinematics giving as simple a description of physical phenomena as possible. Also the coordinates should have the proper metrical significance. Among relevant phenomena is the result of a recent experiment performed by Phipps (1974), the Sagnac effect (Post, 1967), the Lorentz contraction of (corrected) standard measuring rods, and the time dilation of moving standard clocks.

In analogy with the Lorentz transformation, Franklin (1922), Trocheries (1949), and Takeno (1952) have proposed the following transformation to a rotating frame:

$$r'\theta' = \gamma(r\theta - vt), \qquad r' = r, \qquad z' = z, \qquad t' = \gamma\left(t - \frac{vr\theta}{c^2}\right),$$
$$\gamma = \left(1 - \frac{v^2}{c^2}\right)^{-1/2} \tag{3.1}$$

with

$$v = c \tanh(\omega r/c) \tag{3.2}$$

As pointed out by Phipps, the velocity distribution given in equation (3.2) is contrary to the results of his experiment.

Strauss has used the transformation (3.1), but with the linear velocity distribution

$$v = \omega r \tag{3.3}$$

As to the time parameter, he writes.

A correct treatment of rotating frames must use *local times* depending on the radius, because standard clocks revolving in different circles and hence with different speeds cannot be synchronized to show the same time all the time.

From the transformation (3.1) it is apparent that Strauss has used a time parameter measured on standard clocks synchronized according to the

Einstein convention. The transformation is stated to be valid in a cylindrical local subframe with a certain radius r.

However, it may be shown that it is impossible to obtain a consistent synchronization of clocks along a circle around the origin in a rotating frame (Grøn, 1975). Thus the time parameter used in the transformation (3.1) is not well defined. This ambiguity in the chosen time parameter may be removed (Arzeliès, 1956). The practical use of this coordinate system may, however, easily lead to wrong conclusions.

Calculating the velocity of light on a rotating disk in these coordinates, one finds that it is constant and equal to c. This is a direct consequence of the assumed procedure for synchronization of the clocks used. From this result it follows that the time taken by light signals in making complete turns in opposite directions on the rotating disk are equal and independent of the angular velocity of the disk. This, however, is in contradiction with the result of Sagnac's experiment.

Using the Galilean-like transformation

$$\theta' = \theta - \omega t, \qquad r' = r, \qquad z' = z, \qquad t' = t \tag{3.4}$$

one deduces for the velocity of light along the circumference of the disk (Grøn, 1975)

$$c_t = (1 - \omega^2 r^2/c^2)^{-1/2}(c - \omega r) \tag{3.5}$$

The difference in time for the two signals is then, to the first order in $\omega r/c$,

$$\Delta t = 4\omega\pi r^2/c^2 \tag{3.6}$$

in accordance with the result of Sagnac's experiment.

It should be noted that there is no necessity of using standard clocks in a physical description inside a rotating reference frame. The transformation (3.4) implies the use of coordinate clocks that are synchronized and that have their rates adjusted to read the same time as that of the clocks in K. This may be practically performed using time signals from the center of K' (Grøn, 1975).

Strauss states that the spatial coordinates introduced by equation (3.4) have no metrical significance. In consequence of the results in Section 2 this does not seem to be correct. The spatial coordinates with global metrical significance for the rotating disk are a system of measuring rods rigidly fastened to the disk. As different from the standard rods, the coordinate rods are not assumed to perform Born rigid motions. Instead they are assumed to be parts of the physical frame of reference (Brillouin, 1970) for which they have metrical significance. For example, coordinate scales, or a coordinate network, is engraved in the material of the disk. The coordinate transformation $\theta' = \theta - \omega t$ is now seen to be a consequence of the result in Section 2, that there is no Lorentz contraction of the circumference of the disk.

4. *The Spatial Geometry on a Rotating Disk*

Assume that the disk is covered with a network of polar coordinates. The distances between the radii and the circles are so chosen that an arbitrarily given square at the circumference of the disk may be regarded as quadratic when the disk is not rotating.[2] This square defines the unit coordinate length in the radial and the tangential directions. Now this definition is still valid when the disk rotates. The sides of the square define the metrical significant length units on the disk. Thus, relative to the spatial geometry on the disk, the square is still quadratic when the disk rotates.

Euclidean geometry implies that the sides of a square, which is said to be quadratic, have equal lengths as measured with (corrected) *standard* measuring rods, instantaneously at rest relative to the measuring object. Use of such measuring rods gives the result that the lengths of the sides of the "quadratic" square on the rotating disk are unequal. From this one concludes that the geometry of the rotating disk is non-Euclidean.

In view of the arbitrary smallness of the square it is tempting to conclude, as Strauss does, that the geometry of the rotating disk is non-Euclidean, even locally. This conclusion will now be discussed.

Take the Earth as an illustrating example. The surface of the Earth is non-Euclidean globally (spherical) and Euclidean locally (plane). The non-Euclidean character of the surface of the Earth is manifested by the impossibility of covering it with a network of quadratic squares. Regardless of how small the squares are made, those near the poles are far from quadratic. Nevertheless the surface of the Earth is locally Euclidean at the poles. This is evident from the possibility of introducing Euclidean coordinates locally, everywhere on the surface of the earth.

In the same way, the geometry on the rotating disk is globally non-Euclidean. This is manifested by the impossibility of introducing Euclidean (or Minkowskian) coordinates globally on a rotating disk. Near the circumference of the disk, where $\omega r \to c$ when the angular velocity of the disk is made sufficiently great, the squares of the polar-coordinate network chosen above are far from quadratic, measured with standard measuring rods, independently of how small the squares are made. However, it is everywhere possible to introduce inertial frames with Minkowskian coordinates and Euclidean spatial coordinates. By this criterion the spatial geometry of the rotating disk is locally Euclidean.

The spatial geometry in any frame of reference may be mathematically described by introducing the metrically significant coordinates and calculating the form of the proper spatial line element in these coordinates. This is the method used by Møller (1952, p. 238) in obtaining the intrinsic spatial geometry in any frame.

One can define the proper spatial line element by the following operations. Use the radar method and measure the time $d\tau$ taken by a light signal between

[2] Alternatively one could use Cartesian coordinates, covering the disk with a network (but not a whole number) of exactly quadratic squares.

emission and absorption on a standard clock. The proper spatial line element in the direction a between the clock and the reflector is then given by

$$d\sigma_a = \tfrac{1}{2} c\, d\tau \tag{4.1}$$

Using this method it can be shown that the proper spatial line element is generally given by (Landau and Lifshitz, 1971)

$$d\sigma^2 = (g_{ab} - g_{a4} g_{b4}/g_{44})\, dx^a\, dx^b, \qquad a, b = 1, 2, 3 \tag{4.2}$$

where $g_{\mu\nu}$ are the elements of the metric tensor. Møller (1952, Appendix 4) has given the mathematical proof that this element is invariant under a transformation connecting two different coordinate systems inside the same system of reference, and that

$$\gamma_{ab} = g_{ab} - g_{a4} b_{b4}/g_{44} \tag{4.3}$$

transforms as a tensor. The spatial tensor γ_{ab} characterizes the spatial geometry relative to the chosen coordinates. When these are those of the metrically significant coordinate system in a given reference frame, the spatial tensor γ_{ab} characterizes the spatial geometry inside that reference frame.

The method is valid in obtaining both the global and the local geometry in a given reference frame. However, it is essential to distinguish between these cases when coordinates are introduced, and when the mathematical result, given in equation (4.2), is to be interpreted.

Generally the coordinates with local metrical significance are the Minkowskian ones, defined by use of inertial standard measuring rods and clocks, instantaneously at rest relative to the measuring object. Infinitesimal Minkowskian coordinates are generally integrable only in inertial frames. Thus in non inertial frames, of which the rotating disk is an example, the Minkowski coordinates have no metrical significance for the global geometry, and the coordinate system with global metrical significance has no local metrical significance.

The four-dimensional line element

$$ds^2 = g_{\mu\nu} dx^\mu\, dx^\nu, \qquad \mu, \nu = 1, 2, 3, 4 \tag{4.4}$$

may be written (Møller, 1952, p. 245)

$$ds^2 = d\sigma^2 - \left[\frac{g_{a4}}{(-g_{44})^{1/2}} \frac{dx^a}{c\,dt} - (-g_{44})^{1/2}\right]^2 c^2\, dt^2 \tag{4.5}$$

The velocity of a particle is defined by

$$u = \frac{d\sigma}{dt} \tag{4.6}$$

The velocity of light, w, is given by equations (4.5) and (4.6) with $ds = 0$. This gives

$$w = \left[\frac{g_{a4}}{(-g_{44})^{1/2}}\right] \frac{dx^a}{dt} - c(-g_{44})^{1/2} \tag{4.7}$$

The velocity of a particle moving in the direction a follows using equations (4.2) and (4.6):

$$u_a = \frac{d\sigma_a}{dt} = \left(\frac{g_{aa} - g_{a4}^2}{g_{44}}\right)^{1/2} \frac{dx^a}{dt} \tag{4.8}$$

Equations (4.7) and (4.8) lead to the following expression for the velocity of light in the direction a:

$$w_a = \frac{(-g_{44})^{1/2}}{1 + g_{a4}/(g_{a4}^2 - g_{aa}g_{44})^{1/2}}\, c \tag{4.9}$$

Substituting equation (4.7) in equation (4.5) the four-dimensional line element can be written

$$ds^2 = d\sigma^2 - w^2\, dt^2 \tag{4.10}$$

If the coordinates are chosen so that the spatial interval and the velocity of light are directed along one of the coordinate axes, the element may be written

$$ds^2 = d\sigma_a{}^2 - w_a{}^2\, dt^2 \tag{4.11}$$

where w_a is given by equation (4.9).

Strauss criticizes the use of Galilean-like coordinates together with equation (4.2) to obtain the intrinsic spatial geometry in K'. He writes

> The mistake, repeated over and over again, lies in the method used, viz., the use of a Galilean transformation for the introduction of coordinates in the rotating frame. ··· In fact, the method described gives not only wrong results, but leads to inconsistencies: when applied to the transformation between inertial frames it yields non-Euclidean geometries for all inertial frames except the original one.

The proof of this given by Strauss is straightforward and correct. However, it does not concern the relativistic kinematics in rotating frames. If one calculates the velocity of light along the x axis and y axis, employing equation (4.9) with Galilean coordinates, in an inertial system moving along the positive x direction with velocity v, relative to the inertial system in which the velocity of light is isotropic and equal to c, the result is

$$w_x = (1 - v^2/c^2)^{-1/2}(c - v), \qquad w_y = (1 - v^2/c^2)^{1/2} c \tag{4.12}$$

Thus the velocity of light is anisotropic in this description, which is contrary to the results of the Michelson-Morley experiment.

The essential difference between inertial frames and rotating frames is the impossibility of measuring the absolute translational velocity of an inerital system, and the possibility of measuring the angular velocity of the rotating frame. It is a consequence of this difference that the kinematics inside these two types of frames are different. Inside K the velocity of light is isotropic, leading to the use of the Lorentz transformations between inertial frames. In K', however, the result of Sagnac's experiment implies an anisotropic light velocity, which leads one to adopt a Galilean-like transformation from K to K'.

This, however, does not mean the introduction of a prerelativistic, Galilean kinematics in K'. Using the transformation (3.4) and equation (4.2) the proper spatial line element in K' is found to be

$$d\sigma^2 = dr^2 + dz^2 + (1 - \omega^2 r^2/c^2)^{-1} r^2 \, d\theta'^2 \tag{4.13}$$

characterizing a non-Euclidean geometry. The relativistic character of this result is evident from the fact that it can be derived from the relativistic properties of (corrected) standard measuring rods rotating with the disk, and the kinematic properties of the disk, as discussed in Section 2.

The method leading to equation (4.2) for calculating the intrinsic geometry in any frame is essentially a relativistic one, since it is based on the frame invariance of the four-dimensional line element ds^2. Now Strauss states that "the relativistic invariance of ds^2 is a *weaker* postulate than the separate invariances of $d\sigma^2$ and dt^2 postulated by Galilean kinematics." If this statement is intended to mean that the invariance of $d\sigma^2$ and of dt^2 implies the invariance of ds^2, it is not correct. If correct it would mean that the Galilean kinematics implies the invariance of ds^2. This, however, is not the case. The reason is seen in equation (4.11). Even if $d\sigma^2$ and dt^2 are frame invariant in the Galilean kinematics, the velocity of light is not, and this destroys the invariance of ds^2. Thus the invariance of ds^2 distinguishes the relativistic kinematics from the Galilean one.

5. *Conclusion*

The measuring rods moving rigidly on the circumference of a rotating disk are Lorentz contracted. However, it is shown that a transition of the disk from rest to rotational motion, while the circumference satisfies Born's definition of rigidity, is a kinematical impossibility. The elements of the circumference are not Lorentz contracted.

The physical phenomena connected with rotating frames are different from those in inertial frames. In K' the Sagnac effect leads to an anisotropic velocity of light and to the possibility of measuring the absolute angular velocity of the frame. Furthermore, by use of the relativistic kinematics one concludes that synchronization of clocks around a circle about the axis in K', according to the Einstein convention, is impossible.

Taking advantage of these facts, wishing a simple and consistent kinematics in K', and demanding a coordinate system with global metrical significance in K', one is led to the Galilean-like transformation from K to K' given by equation (3.4). When these coordinates are employed in the method used by Møller in obtaining the global intrinsic spatial geometry of the rotating disk, their metrical significance secures that the method is adequate. In this way one is led to the result that the *global* spatial geometry is non-Euclidean inside a rotating reference frame, in accordance with the result of informal arguments based explicitly on the relativistic kinematics. Recognizing the possibility of introducing Minkowskian coordinates locally in the rotating frame, one is led, by the same method, to the result that the *local* spatial geometry is Euclidean inside a rotating reference frame.

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